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2003 J. Phys. A: Math. Gen. 36 4987

(http://iopscience.iop.org/0305-4470/36/18/304)

J. Phys. A: Math. Gen. **36** (2003) 4987–4992 PII: S0305-4470(03)60661-X

Normally ordered expansion of the inverse of the coordinate operator and the momentum operator

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Received 10 March 2003
Published 23 April 2003
Online at stacks.iop.org/JPhysA/36/4987

Abstract
We derive the normally ordered expansion formulae of $\frac{1}{\hat{X}^n}$ and $\frac{1}{\hat{P}^n}$ by virtue of the method of integral within an ordered product of operators in the sense of the principal value integral, where $\hat{X}$, $\hat{P}$ are the coordinate and momentum operator, respectively. Application of the new formula is briefly discussed.

PACS numbers: 02.30.Tb, 03.65.Fd

1. Introduction

It is well known that the normally ordered expansion of operators is very useful in calculating their coherent state expectation values [1]. In [2] and [3] we have proposed the method of integral within an ordered product (IWOP) of operators, by which one can derive the normal product form of many operators in a neat and easy way. For example, we have shown the normally ordered expansion of $\exp(f\hat{X}^2)$, where $\hat{X}$ is the coordinate operator, $\hat{X} = (a + a^\dagger)/\sqrt{2}$, $[a, a^\dagger] = 1$ is the basic bosonic commutator. Using the Fock space expansion of the coordinate eigenvector

$$|x\rangle = \pi^{-\frac{1}{4}} \exp\left[-\frac{x^2}{2} + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right]|0\rangle \qquad \hat{X}|x\rangle = x|x\rangle \tag{1}$$

as well as the normal ordering of the vacuum state projector

$$|0\rangle\langle 0| = :\mathrm{e}^{-a^\dagger a}: \tag{2}$$

and the completeness relation [4]

$$\int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle\langle x| = 1 \tag{3}$$

we obtain

$$\exp(\lambda\hat{X}^2) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\lambda x^2}|x\rangle\langle x| = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\sqrt{\pi}} :\exp[-(x-\hat{X})^2 + \lambda x^2]:$$
$$= :\exp\left[\frac{\lambda}{1-\lambda}\hat{X}^2\right]\Big/\sqrt{1-\lambda}: \qquad \mathrm{Re}\,\lambda < 1. \tag{4}$$

This method is also quite useful in other aspects: for instance, it enables one to directly derive the normally ordered single-mode squeezing operator by simply performing the following integral:

$$\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty} \mathrm{d}x|x/\mu\rangle\langle x| \tag{5}$$

where μ is the squeezing parameter. In this work, we shall derive the normally ordered expansion of the operator $\hat{X}^{-n}$, in particular the one-dimensional Coulomb-like potential $\hat{X}^{-1}$. In the literature before, only the result

$$\frac{1}{\hat{X}} = \frac{1}{\mathrm{i}h}\int_{-\infty}^{p} \mathrm{d}p' \tag{6}$$

was given, since $\hat{X} \to \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p}$ (a derivative in the momentum representation $\langle p|$), so $\frac{1}{\hat{X}}$ is an integral operation in the same representation. But to our knowledge, how to deduce the normally ordered expansion of $\hat{X}^{-1}$ has not been reported before. In addition to the lack of an efficient way to carry out normal ordering, the difficulty may also lie in a singularity appearing in $\hat{X}^{-1}|x=0\rangle$. To circumvent this difficulty, in this work we shall make use of the general definition of inverse operators that successively acting on the original operator and its inverse operator should be equal to an identity operator. So, we expect that the identity $\hat{X}\frac{1}{\hat{X}} = \frac{1}{\hat{X}}\hat{X} = 1$ should serve as a criterion to check the validity of our normally ordered expansion of $\frac{1}{\hat{X}}$. Indeed, as we shall see later, an explicit calculation of $\hat{X}\frac{1}{\hat{X}}$ and $\frac{1}{\hat{X}}\hat{X}$ in the normally ordered form is made to prove that our result does satisfy this criterion and hence justifies its validity. After many attempts in derivation, we find that it is the principal value integral and the IWOP technique that can yield the correct result. The use of principal value integral here is analogous to its wide use in the calculation of various propagators in quantum field theory.

2. Normal ordering expansion of $\frac{1}{\hat{X}}$ gained via principal value integral within ordered product of operators

By using (3) we may encounter the integral

$$\frac{1}{\hat{X}} = \int_{-\infty}^{+\infty} \mathrm{d}x\frac{1}{x}|x\rangle\langle x|. \tag{7}$$

Strictly speaking, a mathematical integral with a pole on the contour does not exist. Yet one may recall that in deriving various propagators in quantum field theory people successfully used the principal value integral and contour integral prescription when they met singularities and checked the outcome to ensure its validity and agreement with boundary conditions. Thus we are motivated to do the same thing here. For a simple pole at $x=0$ on the real axis in (7), we define the Cauchy principal value integral, i.e.

$$\frac{1}{\hat{X}} = :\lim_{A\to\infty,\epsilon\to 0+} \frac{1}{\sqrt{\pi}}\int_{-A}^{-\epsilon} \mathrm{d}x\frac{\exp[-(x-\hat{X})^2]}{x} + \frac{1}{\sqrt{\pi}}\int_{\epsilon}^{A} \mathrm{d}x\frac{\exp[-(x-\hat{X})^2]}{x}:. \tag{8}$$

As we shall check later, the normally ordered expansion obtained integral of (8) does satisfy the criterion $\hat{X}\frac{1}{\hat{X}} = \frac{1}{\hat{X}}\hat{X} = 1$. Letting $x=-t$ in the first term, equation (8) becomes

$$\frac{1}{\hat{X}} = :\lim_{A\to\infty,\epsilon\to 0+} \frac{1}{\sqrt{\pi}}\int_A^\epsilon \mathrm{d}t \frac{\exp[-(t+\hat{X})^2]}{t} + \frac{1}{\sqrt{\pi}}\int_\epsilon^A \mathrm{d}x \frac{\exp[-(x-\hat{X})^2]}{x} :$$
$$= :\lim_{A\to\infty,\epsilon\to 0+} \frac{1}{\sqrt{\pi}}\int_\epsilon^A \mathrm{d}x \frac{-\exp[-(x+\hat{X})^2]+\exp[-(x-\hat{X})^2]}{x} :$$
$$= :\frac{1}{\sqrt{\pi}}\int_0^{+\infty} \mathrm{d}x\, \mathrm{e}^{-x^2} \frac{\exp[2x\hat{X}] - \exp[-2x\hat{X}]}{x} \exp[-\hat{X}^2] : . \tag{9}$$

Now the integral converges at both $x = 0$ and $x \to +\infty$. It then follows:

$$\frac{1}{\hat{X}} = :\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} \frac{2^{2k+2}}{(2k+1)!}\left(\int_0^{+\infty}\mathrm{d}x\,\mathrm{e}^{-x^2}x^{2k}\right)\hat{X}^{2k+1}\exp[-\hat{X}^2] :$$
$$= :\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!}\left(\int_0^{+\infty}\mathrm{d}t\,\mathrm{e}^{-t}t^{k-\frac{1}{2}}\right)\hat{X}^{2k+1}\exp[-\hat{X}^2] :$$
$$= :\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!}\Gamma\left(k+\frac{1}{2}\right)\hat{X}^{2k+1}\exp[-\hat{X}^2] : \tag{10}$$

where we have used the well-known formula

$$\int_0^{+\infty}\mathrm{d}t\,\mathrm{e}^{-t}t^{\nu-1} = \Gamma(\nu) \qquad \mathrm{Re}(\nu) > 0. \tag{11}$$

Employing the combinatorial formula [5] (see appendix A)

$$\sum_{k=0}^{n}\frac{x}{k+x}(-1)^k\binom{n}{k} = \frac{1}{\binom{n+x}{n}} \qquad \binom{\lambda}{k} = \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!} \qquad k \text{ is an integer} \tag{12}$$

and $\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}2^{-k}(2k-1)!!$ we can further simplify (10) as

$$\frac{1}{\hat{X}} = :\sum_{k=0}^{\infty}\frac{2}{(2k+1)k!}\hat{X}^{2k+1}\sum_{m=0}^{\infty}\frac{(-1)^m\hat{X}^{2m}}{m!} :$$
$$= :\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{2(-1)^{n-k}}{(2k+1)k!(n-k)!}\hat{X}^{2n+1} :$$
$$= :2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\frac{1}{2}}{k+\frac{1}{2}}(-1)^k\binom{n}{k}\right)\frac{(-1)^n}{n!}\hat{X}^{2n+1} :$$
$$= :2\sum_{n=0}^{\infty}\frac{1}{\binom{n+1/2}{n}}\frac{(-1)^n}{n!}\hat{X}^{2n+1} : \tag{13}$$

which is neat and concise. Or we can rewrite (13) as

$$\frac{1}{\hat{X}} = :2\sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\cdots\frac{3}{2}}\hat{X}^{2n+1} := :\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)}\hat{X}^{2n+1} : . \tag{14}$$

From equation (14) we see that the operator $\frac{1}{\hat{X}}$ is not ill-defined in Fock space as one might think. Operating $\frac{1}{\hat{X}}$ directly on the vacuum state we have

$$\frac{1}{\hat{X}}|0\rangle = 2\sum_{n=0}^{\infty}\frac{1}{\binom{n+1/2}{n}\binom{n+1/2}{n}}\frac{(-1)^n}{n!}(a^\dagger/\sqrt{2})^{2n+1}|0\rangle = \sqrt{2}\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{(2n)!!}{(2n+1)!!}}|2n+1\rangle \tag{15}$$

which is a superposition of odd number states.

3. Checking the validity of (15)

In order to confirm the above result we multiply the two sides of (14) by $\hat{X}$, or $\frac{1}{\sqrt{2}}(a+a^\dagger)$,

$$\begin{aligned}\hat{X}\frac{1}{\hat{X}} &= \sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}\{a : (a+a^\dagger)^{2n+1} : + : a^\dagger(a+a^\dagger)^{2n+1} :\}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}\left\{: (a+a^\dagger)^{2n+2} : + : (2n+1)(a^\dagger+a)^{2n} :\right\}\\ &= :\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}(a+a^\dagger)^{2n+2} : + : \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!!}(a^\dagger+a)^{2n} : +1 = 1\end{aligned} \tag{16}$$

where we have used the commutator

$$[a, : f(a, a^\dagger) :] = :\frac{\partial}{\partial a^\dagger}f(a, a^\dagger) : \tag{17}$$

with $: f(a, a^\dagger) :$ taking the particular function $: (a+a^\dagger)^{2n+1} :$. Similarly, using the commutator

$$[: f(a, a^\dagger) :, a^\dagger] = :\frac{\partial}{\partial a}f(a, a^\dagger) : \tag{18}$$

we can confirm $\frac{1}{\hat{X}}\hat{X} = 1$. In this way, we have checked the validity of the normally ordered expansion of the inverse of coordinate operator.

4. Normally ordered expansion of $\hat{X}^{-n}$

To derive the normally ordered expansion of $\hat{X}^{-n}$, we note that there exists an identity [6]

$$\frac{\mathrm{d}}{\mathrm{d}\hat{X}} : \hat{X}^n : = : \frac{\mathrm{d}}{\mathrm{d}\hat{X}}\hat{X}^n : \tag{19}$$

(see appendix B). Hence we can obtain the normally ordered expansion of higher power of $\frac{1}{\hat{X}}$,

$$\begin{aligned}\frac{1}{\hat{X}^n} &= \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}\hat{X}}\right)^{n-1}\frac{1}{\hat{X}}\\ &= \sqrt{\pi}\frac{(-1)^{n-1}}{(n-1)!} : \sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma\left(m+\frac{3}{2}\right)}\left(\frac{\mathrm{d}}{\mathrm{d}\hat{X}}\right)^{n-1}\hat{X}^{2m+1} :\\ &= \sqrt{\pi}(-1)^{n-1} : \sum_{m=\frac{[n-1]}{2}}^{\infty}\frac{(-1)^m}{\Gamma\left(m+\frac{3}{2}\right)}\binom{2m+1}{n-1}\hat{X}^{2m-n+2} :\\ &= \sqrt{\pi}(-1)^n : \sum_{m=\frac{[n-1]}{2}}^{\infty}\frac{(-1)^m}{\Gamma\left(m+\frac{1}{2}\right)}\binom{2m-1}{n-1}\hat{X}^{2m-n} : .\end{aligned} \tag{20}$$

Comparing (20) with the normally ordered expansion of $\hat{X}^n$

$$\hat{X}^n = \frac{1}{\sqrt{\pi}} \sum_{l=0}^{[n/2]} \Gamma\left(l + \frac{1}{2}\right) \binom{n}{2l} : \hat{X}^{n-2l} :$$

we find that they bear some formal correspondence corresponding to $n \to -n$. In particular, when $n = 2$,

$$\frac{1}{\hat{X}^2} = \sqrt{\pi} : \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(2m+1)}{\Gamma\left(m + \frac{3}{2}\right)} \hat{X}^{2m} : . \tag{21}$$

5. Application of (15)

Here we consider a harmonic oscillator perturbed by a potential $\frac{1}{\hat{X}}$,

$$H = \frac{P^2}{2} + \frac{1}{2}\hat{X}^2 + \lambda \frac{1}{\hat{X}} \tag{22}$$

where λ is small. As one can see from (14) that $\frac{1}{\hat{X}}$ is not ill-defined in Fock space, so this Hamiltonian makes sense in Fock space too. If the unperturbed state is in a coherent state

$$|z\rangle = \exp\left[za^\dagger - \frac{1}{2}|z|^2\right]|0\rangle \tag{23}$$

then

$$\langle z|\frac{1}{\hat{X}}|z\rangle = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1/2}\Gamma\left(n + \frac{3}{2}\right)} (z + z^*)^{2n+1}. \tag{24}$$

In summary, we have derived the normally ordered expansion formula of $\frac{1}{\hat{X}^n}$ by virtue of the IWOP technique in the sense of the principal value integral, which are the supplement of the expansion of $\hat{X}^n$. Needless to say, one can obtain the normally ordered expansion of $\frac{1}{\hat{P}^n}$ in the same way,

$$\frac{1}{\hat{P}} = : 2 \sum_{n=0}^{\infty} \frac{1}{\binom{n+1/2}{n}} \frac{(-1)^n}{n!} \hat{P}^{2n+1} : . \tag{25}$$

Acknowledgment

We are very much indebted to the editorial board member for his fair comment on this work. This work is supported by the President Foundation of Chinese Academy of Science and National Natural Science Foundation of China under grant 10175057.

Appendix A

Using the induction method, formula (12) can be proved as follows: letting $T_n \equiv \sum_{k=0}^{n} \frac{x}{k+x} (-1)^k \binom{n}{k}$, we have $T_0 = 1$,

$$\begin{aligned} T_n &= 1 + \sum_{k=1}^{n} \frac{x}{k+x} (-1)^k \binom{n}{k} \\ &= 1 + \sum_{k=1}^{n-1} \frac{x}{k+x} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) + (-1)^n \frac{x}{n+x} \end{aligned}$$

$$= T_{n-1} + \frac{x}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{k}{k+x}$$
$$= T_{n-1} + \frac{x}{n}\left\{\sum_{k=1}^{n}(-1)^k\binom{n}{k} - \sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{x}{k+x}\right\}$$
$$= T_{n-1} - \frac{x}{n}T_n. \tag{26}$$

Therefore, we immediately obtain the recurrence relation,

$$T_n = \frac{n}{n+x}T_{n-1} = \frac{n}{n+x}\frac{n-1}{n+x-1}T_{n-2} = \cdots = \frac{n!}{(n+x)(n+x-1)\cdots(x+1)}T_0 = \frac{1}{\binom{n+x}{n}}. \tag{27}$$

Appendix B

To prove equation (19), we recall the mathematical formula [7]

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\pi}\exp[-(x-y)^2]H_n(x) = (2y)^n \tag{28}$$

where H_n denotes the nth Hermite polynomials. It then follows:

$$H_n(\hat{X}) = \int_{-\infty}^{\infty}\mathrm{d}x|x\rangle\langle x|H_n(x)$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\pi} : \exp[-(x-\hat{X})^2]H_n(x) : = 2^n : \hat{X}^n : . \tag{29}$$

Compare it with the well-known recurrence relation of the Hermite polynomials,

$$H_n^{'}(x) = 2nH_{n-1}(x) \tag{30}$$

we see

$$\frac{\mathrm{d}}{\mathrm{d}\hat{X}} : \hat{X}^n : = 2^{-n}\frac{\mathrm{d}}{\mathrm{d}\hat{X}}H_n(\hat{X}) = n2^{1-n}H_{n-1}(\hat{X}) = n : \hat{X}^{n-1} : = : \frac{\mathrm{d}}{\mathrm{d}\hat{X}}\hat{X}^n :$$

thus (19) really holds.

References

[1] Glauber R J 1963 *Phys. Rev.* **130** 1529
[2] Fan H-y, Zaidi H R and Klauder J R 1987 *Phys. Rev.* **35** 1831
Fan H-y and Chen J-h 2001 *J. Phys. A: Math. Gen.* **34** 10939
Fan H-y and Fu L *J. Phys. A: Math. Gen.* **36** 1531–6
[3] Wü 1999 *J. Opt. B: Qauntum Semiclass. Opt.* **1** R11
[4] Dirac P A M 1958 *Principles of Quantum Mechanics* 3rd edn (Oxford: Oxford University Press)
[5] Gould H W 1972 *Combinatorial Identities* (Morgantown, West Virginia)
[6] Fan H-y and Fan Y 2000 *Commun. Theor. Phys.* **34** 149
[7] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* ed Alan Jeffery (New York: Academic)